WEAK HARMONIC MAASS FORMS AND MOCK MODULAR FORMS

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Abstract. These are the notes of a talk at the Mock Modular Forms seminar at Concordia University, 5 October 2017. We define weak harmonic Maass forms and mock modular forms and give some examples. We then prove surjectivity of the shadow map ξ. We mainly follow the expositions in [BF] and [Ono]. No originality is claimed and any mistake found here is due to the author.

Contents

1. Definitions

1.1. **First definitions.** Let $k \in \frac{1}{2}\mathbf{Z}$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ ($\gamma \in \Gamma_0(4)$ if $k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$). Define the automorphy factor for $\tau \in \mathcal{H}$ and γ by

$$j(\gamma, \tau) = \begin{cases} \sqrt{c\tau + d} & \text{if } k \in \mathbf{Z} \\ \left(\frac{c}{d}\right) \epsilon_d^{-1} \sqrt{c\tau + d} & \text{if } k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z} \end{cases}$$

where $\left(\frac{c}{d}\right)$ denotes the Jacobi symbol and $\epsilon_d = 1$ or i depending on whether $d = 1$ or 3 modulo 4. We always take the principal branch of the square root. We remark that if $k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$, then $j(\gamma, \tau)$ is the automorphy factor of the theta function $\Theta(\tau) = \sum_{n \in \mathbf{Z}} q^{n^2}$. We define the weight k slash operator on functions $f : \mathcal{H} \longrightarrow \mathbf{C}$ by $(f|_k\gamma)(\tau) = j(\gamma, \tau)^{-2k} f(\gamma\tau)$.

Definition 1.1. Let $N \in \mathbf{N}$, $k \in \frac{1}{2}\mathbf{Z}$ ($4|N$ if $k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$) and let χ be a Dirichlet character of modulus N. A weak harmonic Maass form of weight k, level N and Nebentypus χ is a smooth function $f : \mathcal{H} \longrightarrow \mathbf{C}$ satisfying:

(a) $f|_k\gamma = \chi(d) f$ for all $\gamma \in \Gamma_0(N)$.
(b) For each cusp $s = \alpha(\infty)$ of $\Gamma_0(N)$, we have the following growth condition

$$(f|_k\alpha)(\tau) \ll e^{Cy} \text{ as } y \to +\infty$$

uniformly in x where $\tau = x + iy$ and for some $C > 0$.

(c) $\Delta_k f = 0$ where Δ_k is the hyperbolic Laplacian of weight k defined by

$$\begin{aligned}\Delta_k &= -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})\\ &= -4y^2\frac{\partial^2}{\partial\tau\partial\bar{\tau}}+2iky\frac{\partial}{\partial\bar{\tau}}.\end{aligned}$$

The space of such functions is denoted by $H_k(N,\chi)$.

Notation. We shall write $M_k^!(N,\chi)$, $M_k(N,\chi)$ and $S_k(N,\chi)$ respectively for the weakly holomorphic, the holomorphic and the cuspidal modular forms of weight k, level N and Nebentypus χ. If the character is trivial we shall omit it in the notation.

Remark 1.2. The space $M_k^!(N,\chi)$ forms a subspace of $H_k(N,\chi)$ since holomorphic functions are by definition annihilated by the anti-holomorphic derivative.

1.2. **Fourier expansions.** Like holomorphic modular forms, weak harmonic Maass forms admit Fourier expansions that we shall now derive. We will suppose here and in the rest of these notes that $k\neq 1$. Weight 1 forms also admit Fourier expansions with a slight modification to the coefficients which can be derived in a similar way to what follows.

Let $f\in H_{2-k}(N,\chi)$ with $k\neq 1$. Fix $y>0$ and consider the function $f_y:\mathbf{R}\longrightarrow\mathbf{C}$ defined by $f_y(x)=f(x+iy)$. This function is 1-periodic by (a) and by smoothness admits a Fourier expansion which converges absolutely and uniformly

$$f(\tau)=f_y(x)=\sum_{n\in\mathbf{Z}}a_n(y)q^n=\sum_{n\in\mathbf{Z}}a_n(y)e^{-2\pi ny}e(nx)$$

where $\tau=x+iy$, $e(u)=e^{2\pi iu}$ and $q=e(\tau)$. By uniform convergence and (c) the coefficients of this series satisfy the differential equation

$$\Delta_k(an(y)q^n)=0\Leftrightarrow\frac{\mathrm{d}^2a_n}{\mathrm{d}y^2}(y)=\left(4\pi n+\frac{k-2}{y}\right)\frac{\mathrm{d}a_n}{\mathrm{d}y}(y)$$

which is solved by successively solving two degree 1 equations, the first one giving an expression for $\frac{\mathrm{d}a_n}{\mathrm{d}y}$ and the second one yielding an expression for a_n. This produces the following solutions

$$\begin{cases}a_n(y)=c_f^+(n)+c_f^-(n)\int_{-4\pi ny}^{\infty}e^{-t}t^{k-2}\mathrm{d}t & \text{for } n\neq 0\\ a_0(y)=c_f^+(0)+c_f^-(0)y^{k-1}.\end{cases}$$

We define $H(w)=e^{-w}\int_{-2w}^{\infty}e^{-t}t^{k-2}\mathrm{d}t$. This integral converges for $k>1$ and admits a meromorphic (holomorphic if $w\neq 0$) continuation in k in a similar way as for the gamma function. If $w<0$, then $H(w)=e^{-w}\Gamma(k-1,-2w)$ where $\Gamma(s,x):=\int_x^{\infty}e^{-t}t^{s-1}\mathrm{d}t$ is the incomplete gamma function. We have the following asymptotic behaviour

$$H(w)\sim\begin{cases}(2|w|)^{-k}e^{-|w|} & \text{for } w\to-\infty\\ (-2w)^{-k}e^{w} & \text{for } w\to+\infty.\end{cases}$$

This is easily checked by dividing $H(w)$ by these expressions and computing the limits. We have

$$f(\tau)=\sum_{n\in\mathbf{Z}}c_f^+(n)q^n+c_f^-(0)y^{k-1}+\sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}}c_f^-(n)H(2\pi ny)e(nx).$$

Using the asymptotics for the function H we see that the growth condition (b) forces

$$\begin{cases}c_f^+(n)=0 & \text{for all but finitely many } n<0\\ c_f^-(n)=0 & \text{for all but finitely many } n>0.\end{cases}$$

We have thus proved the following result.

Proposition 1.3. *Let $f \in H_{2-k}(N,\chi)$ with $k \neq 1$. At the cusp ∞ the function f admits the Fourier expansion*

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) y^{k-1} + \sum_{\substack{n \ll +\infty \\ n \neq 0}} c_f^-(n) H(2\pi n y) e(nx).$$

A similar result holds at the other cusps.

Definition 1.4. Let $f \in H_{2-k}(N,\chi)$. The holomorphic part of f is defined by

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

and is called a mock modular form of weight k, level N and Nebentypus χ. The non-holomorphic part of f is denoted by

$$f^-(\tau) := c_f^-(0) y^{k-1} + \sum_{\substack{n \ll +\infty \\ n \neq 0}} c_f^-(n) H(2\pi n y) e(nx).$$

1.3. **The operator ξ.** Consider the differential operator $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \bar\tau}}$.

Proposition 1.5. *The above operator defines an anti-linear map*

$$\xi_{2-k} : H_{2-k}(N,\chi) \longrightarrow M_k^!(N,\bar\chi)$$

given by

$$\xi_{2-k} f(\tau) = \xi_{2-k} f^-(\tau) = \overline{c_f^-(0)}(k-1) - (4\pi)^{k-1} \sum_{\substack{n \gg -\infty \\ n \neq 0}} \overline{c_f^-(-n)} n^{k-1} q^n.$$

Proof. Let $f \in H_{2-k}(N,\chi)$. Note that $\Delta_{2-k} = \xi_k \xi_{2-k}$. Thus $\Delta_{2-k} f = 0$ implies that $\xi_k(\xi_{2-k} f) = 0$ which gives $\frac{\partial f}{\partial \bar\tau} = 0$. This proves that $\xi_{2-k} f$ is a holomorphic function on the upper half plane. The modular transformation property can be checked by direct computation. It remains to see that the conditions at the cusps are satisfied in order to prove that $\xi_{2-k} f$ is indeed a weakly holomorphic modular form.

Since f^+ is holomorphic we have $\xi_{2-k} f = \xi_{2-k} f^-$. Writing out the definition of the function H we have

$$f^-(\tau) = c_f^-(0) y^{k-1} + \sum_{\substack{n \ll +\infty \\ n \neq 0}} c_f^-(n) \left(\int_{-4\pi n y}^{\infty} e^{-t} t^{k-2} \mathrm{d}t \right) q^n.$$

A quick computation yields $\xi_{2-k}(y^{k-1}) = k-1$ and $\xi_{2-k}(\int_{-4\pi n y}^{\infty} e^{-t} t^{k-2} \mathrm{d}t) = -(-4\pi n)^{k-1} e^{4\pi n y}$. Whence

$$\xi_{2-k} f^-(\tau) = \overline{c_f^-(0)}(k-1) - (4\pi)^{k-1} \sum_{\substack{n \ll +\infty \\ n \neq 0}} \overline{c_f^-(n)} (-n)^{k-1} e^{4\pi n y} \overline{q^n}.$$

Notice that $e^{4\pi n y} \overline{q^n} = q^{-n}$. By change of variable we get the formula

$$\xi_{2-k} f^-(\tau) = \overline{c_f^-(0)}(k-1) - (4\pi)^{k-1} \sum_{\substack{n \gg -\infty \\ n \neq 0}} \overline{c_f^-(-n)} n^{k-1} q^n.$$

A similar formula holds at each cusp. This proves that $\xi_{2-k} f$ is meromorphic at the cusps and thus belongs to $M_k^!(N,\bar\chi)$ as claimed. □

Remark 1.6. Let $f \in H_{2-k}(N,\chi)$ with $\xi_{2-k} f = 0$. Then $f^- = 0$ so f is holomorphic. Thus the kernel of ξ_{2-k} is $M_{2-k}^!(N,\chi)$.

Definition 1.7. If $f \in H_{2-k}(N,\chi)$, then the form $\xi_{2-k} f \in M_k^!(N,\bar{\chi})$ is called the shadow of f.

Remark 1.8. The coefficients of the non-holomorphic part of $f \in H_{2-k}(N,\chi)$ can be recovered from the shadow of f via a period integral.

Definition 1.9. We define $H_{2-k}^+(N,\chi)$ to be the subspace of $H_{2-k}(N,\chi)$ consisting of weak harmonic Maass forms whose shadow is a cusp form, that is $H_{2-k}^+(N,\chi) := \xi_{2-k}^{-1}(S_k(N,\bar{\chi}))$.

The forms f that belong to the subspace $H_{2-k}^+(N,\chi)$ satisfy the following growth condition at the cusps:

(b') For each cusp $s = \alpha(\infty)$ of $\Gamma_0(N)$ there exists a polynomial $P_{s,f} \in \mathbf{C}[q^{-1}]$ such that

$$(f|_{2-k}\alpha)(\tau) - P_{s,f}(\tau) \ll e^{-\epsilon y} \text{ as } y \to +\infty \text{ for some } \epsilon > 0.$$

These forms are completely determined by this condition in the sense that we could define $H_{2-k}^+(N,\chi)$ to be the space of smooth functions $f : \mathcal{H} \longrightarrow \mathbf{C}$ satisfying conditions $(a), (b')$ and (c). This is the definition of weak harmonic Maass forms given in [Ono] whereas the definition that we use in these notes is the more general one of [BF]. Incorporating this stronger growth condition at the cusps in our previous proof of the Fourier expansion yields the following result.

Proposition 1.10. *Let $f \in H_{2-k}^+(N,\chi)$. At the cusp ∞ the function f admits the Fourier expansion*

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1,-4\pi n y) q^n.$$

A similar result holds at the other cusps.

Remark 1.11. Note that if $k \leq 0$, then $S_k(N,\bar{\chi}) = 0$ so that $H_{2-k}^+(N,\chi) = M_{2-k}^!(N,\chi)$ in this case.

2. Examples

a) All weakly holomorphic modular forms are weak harmonic Maass forms as we have already remarked. More precisely we have the inclusions $M_{2-k}^!(N,\chi) \subset H_{2-k}^+(N,\chi) \subset H_{2-k}(N,\chi)$.

b) This example concerns the incoherent Eisenstein series of Kudla, Rapoport and Yang that was mentioned in the introduction talk of this seminar. Let $q > 3$ be a prime congruent to 3 modulo 4, set $K = \mathbf{Q}(\sqrt{-q})$ and let $\chi_q : (\mathbf{Z}/q\mathbf{Z})^\times \to \mathbf{C}^\times$ be a quadratic character. Consider the Eisenstein family

$$E_-(\tau,s) := y^{s/2} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (c\tau+d)^{-1}|c\tau+d|^{-s}\phi_q^-(\gamma)$$

where $\phi_q^-(c,d) = \chi_q^{-1}(d)$ if $c \equiv 0 \mod q$ and $\phi_q^-(c,d) = -iq^{-1/2}\chi_q(c)$ otherwise. The completed Eisenstein series $E_-^*(\tau,s) := q^{s+1/2}\Lambda(s+1,\chi_q)E_-(\tau,s)$ has a zero at $s=0$. Kudla, Rapoport and Yang proved that the incoherent Eisenstein series attached to $(1,\chi_q)$ defined by

$$\phi(\tau) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E_-^*(\tau,s)$$

belongs to $H_1(q,\chi_q)$ and has shadow equal to the theta series θ_K. For more details we refer the reader to [KRY].

c) This example was also mentioned in the introduction talk and concerns the independent work of Duke, Li, Ehlen and Viazovska. Using the same notation as in the previous example, let H denote the Hilbert class field of K and let $\psi : \mathrm{Gal}(H/K) \longrightarrow \mathbf{C}^\times$ denote a class group character. Let θ_ψ denote the theta-series attached to ψ. This is a cusp form of weight 1 if ψ is non-trivial and equal to $E_1(1,\chi_q)$ if ψ is trivial. Duke, Li, Ehlen and Viazovska explicitly construct a weak harmonic Maass form in $H_1(q,\chi_q)$ which has shadow θ_ψ and whose associated mock modular form has coefficients that encode interesting arithmetic information in line with the Duke-Li conjecture.

d) The weight two Eisenstein series is defined by

$$\begin{aligned} E_2(\tau) &= \frac{1}{2\zeta(2)} \sum_{m\in\mathbf{Z}} \sum_{n\in\mathbf{Z}}{}' \frac{1}{(m\tau+n)^2} \\ &= 1 + \frac{3}{\pi^2} \sum_{\substack{m\in\mathbf{Z} \\ n\neq 0}} \sum_{n\in\mathbf{Z}} \frac{1}{(m\tau+n)^2} \\ &= 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n. \end{aligned}$$

The sum in the definition of this function does not converge absolutely. The Eisenstein series is still holomorphic but it is not modular. In fact, we have $E_2(\tau+1) = E_2(\tau)$ but because of the failure of absolute convergence we have $\tau^{-2}E_2(-1/\tau) = E_2(\tau) + \frac{6}{\pi i \tau}$. We introduce a correction $E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi y}$. This function is modular but no longer holomorphic. It turns out that E_2^* belongs to $H_2(\mathrm{SL}_2(\mathbf{Z}))$ and has shadow the constant function $3/\pi$. It follows that the weight two Eisenstein series E_2 is a mock modular form of weight 2 and level 1.

e) This example concerns Zagier's mock modular form. For n positive, let $H(n)$ denote the Hurwitz class number of n, that is the class number of positive definite binary quadratic forms of discriminant $-n$ where forms are weighted by $2/g$ with g the order of their automorphism group. Set $H(0) = -1/12$.

Theorem 2.1 (Zagier)**.** *The function*

$$G(\tau) := \sum_{n\geq 0} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n\in\mathbf{Z}} \beta(4\pi n^2 y) q^{-n^2}$$

belongs to $H_{3/2}(\Gamma_0(4))$ *where* $\beta(s) := \int_1^\infty t^{-3/2} e^{-st} \mathrm{d}t$.

The shadow of G is equal to the Jacobi theta function. In particular, Zagier shows that the generating function of the Hurwitz class number is a mock modular form of weight 3/2.

3. Surjectivity of the operator ξ

In this section we consider $k \in \mathbf{Z}$. We have defined the anti-linear map ξ_{2-k} from $H_{2-k}(N)$ to $M_k^!(N)$ and seen that its kernel is given by $M_{2-k}^!(N)$. The goal of this section is to prove that it is surjective and thus that it induces exact sequences

$$0 \longrightarrow M_{2-k}^!(N) \longrightarrow H_{2-k}(N) \longrightarrow M_k^!(N) \longrightarrow 0$$

and

$$0 \longrightarrow M_{2-k}^!(N) \longrightarrow H_{2-k}^+(N) \longrightarrow S_k(N) \longrightarrow 0.$$

Our main tool in the proof of the surjectivity comes from complex geometry.

3.1. Hodge star operators on $\mathcal{H}$.

The general theory of Hodge star operators and Laplacians on manifolds can be found in [Voi], chapter 5. In this section we specialise to the complex upper half-plane $\mathcal{H}$ which is a complex manifold of real dimension 2. Let $\mathcal{C}^\infty_{\mathbf{R}}$ denote the sheaf of real-valued smooth functions on $\mathcal{H}$ and let $\mathcal{O}_{\mathcal{H}}$ denote the sheaf of holomorphic functions on $\mathcal{H}$. Let $T_{\mathbf{R}}$ denote the real tangent bundle of $\mathcal{H}$ and let $\mathcal{E}^1_{\mathbf{R}}$ denote the sheaf of real-valued differential 1-forms on $\mathcal{H}$. If $\tau \in \mathcal{H}$ then by definition the dual vector space $(T_{\mathbf{R},\tau})^*$ is equal to the fiber $\mathcal{E}^1_{\mathbf{R},\tau}$ of $\mathcal{E}^1_{\mathbf{R}}$ at τ. Consider the Poincaré metric given for $\tau = x + iy \in \mathcal{H}$ on the tangent space $T_{\mathbf{R},\tau}$ by

$$g_\tau = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}.$$

This defines a Riemannian metric on the upper half-plane $\mathcal{H}$.

By Riesz's representation theorem, the linear map $r : T_{\mathbf{R},\tau} \longrightarrow \mathcal{E}^1_{\mathbf{R},\tau}$ defined by $Y_\tau \mapsto g_\tau(\,\cdot\,, Y_\tau)$ is an isomorphism. If $\alpha_\tau \in \mathcal{E}^1_{\mathbf{R},\tau}$, then let $\alpha_\tau^\# := r^{-1}(\alpha_\tau)$. We define an inner product on $\mathcal{E}^1_{\mathbf{R},\tau}$ by

$$\langle \alpha_\tau, \beta_\tau \rangle_\tau := g_\tau(\alpha_\tau^\#, \beta_\tau^\#).$$

One checks easily that $dx^\# = y^2 \frac{\partial}{\partial x}$ and $dy^\# = y^2 \frac{\partial}{\partial y}$ so that the inner product is given by the matrix $\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$. Finally, we endow $\bigwedge^2 \mathcal{E}^1_{\mathbf{R},\tau}$ with the inner product

$$\langle dx \wedge dy, dx \wedge dy \rangle_\tau = \det \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix} = y^4.$$

It follows that the volume form on $\mathcal{H}$ associated to the Poincaré metric is given by $\mathrm{Vol}_\tau = \frac{dx \wedge dy}{y^2}$.

By convention we set $\bigwedge^0 \mathcal{E}^1_{\mathbf{R},\tau} = \mathcal{C}^\infty_{\mathbf{R},\tau}$ and we endow this space with the inner product defined by $\langle f, g \rangle_\tau := f(\tau)g(\tau)$. Let $k \in \{0, 1, 2\}$. Right exterior product gives an isomorphism

$$p : \bigwedge^{2-k} \mathcal{E}^1_{\mathbf{R},\tau} \longrightarrow \mathrm{Hom}\Big(\bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau}, \bigwedge^2 \mathcal{E}^1_{\mathbf{R},\tau}\Big)$$

defined by $\beta_\tau \mapsto \,\cdot\, \wedge \beta_\tau$. Moreover, $\bigwedge^2 \mathcal{E}^1_{\mathbf{R},\tau}$ is trivialised by the volume form Vol_τ. By Riesz's representation theorem, we have an isomorphism

$$m : \bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau} \longrightarrow \Big(\bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau}\Big)^*$$

defined by $\alpha_\tau \mapsto \langle \,\cdot\,, \alpha_\tau \rangle_\tau$. We define the Hodge star at τ to be

$$*_\tau := p^{-1} \circ m : \bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau} \longrightarrow \bigwedge^{2-k} \mathcal{E}^1_{\mathbf{R},\tau}.$$

It is characterised by the property that if $\beta_\tau \in \bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau}$, then for all $\alpha_\tau \in \bigwedge^k \mathcal{E}^1_{\mathbf{R},\tau}$ we have

$$\alpha_\tau \wedge *_\tau \beta_\tau = \langle \alpha_\tau, \beta_\tau \rangle_\tau \mathrm{Vol}_\tau.$$

These maps glue together to form a linear bundle isomorphism called the Hodge star

$$* : \mathcal{E}^k_{\mathbf{R}} \longrightarrow \mathcal{E}^{2-k}_{\mathbf{R}}.$$

Remark 3.1. It can be checked that $*^{-1} = (-1)^{k(2-k)} *$.

We now turn to the computation of the Hodge star. We claim that $*1 = \mathrm{Vol}$. Indeed, for any $f \in \mathcal{C}^\infty_{\mathbf{R}}$ and any τ we have

$$f(\tau) \wedge \mathrm{Vol}_\tau = f(\tau)\mathrm{Vol}_\tau = \langle f, 1 \rangle_\tau \mathrm{Vol}_\tau.$$

We claim that $*dx = dy$. Indeed, we have

$$dx \wedge dy = y^2 \frac{dx \wedge dy}{y^2} = \langle dx, dx \rangle_\tau \mathrm{Vol}_\tau \quad \text{and} \quad dy \wedge dx = 0 = \langle dy, dx \rangle_\tau \mathrm{Vol}_\tau.$$

Similarly one can check that $*dy = -dx$ and $*(dx \wedge dy) = y^2$. We record these results:

$$\begin{aligned} *dx &= dy & *dy &= -dx \\ *1 &= \tfrac{dx \wedge dy}{y^2} & *dx \wedge dy &= y^2. \end{aligned} \tag{1}$$

Remark 3.2. Consider the sheaf $\mathcal{E}^1_{\mathbf{C}}$ of complex valued differential 1-forms on $\mathcal{H}$. We have $\mathcal{E}^1_{\mathbf{C},\tau} = \mathcal{E}^1_{\mathbf{R},\tau} \otimes \mathbf{C}$ and we extend the local Hodge star at τ by $\mathbf{C}$-linearity to an isomorphism

$$*_\tau : \bigwedge^k \mathcal{E}^1_{\mathbf{C},\tau} \longrightarrow \bigwedge^{2-k} \mathcal{E}^1_{\mathbf{C},\tau}.$$

If we extend the inner product on $\mathcal{E}^1_{\mathbf{R},\tau}$ to a Hermitian product on $\mathcal{E}^1_{\mathbf{C},\tau}$ then the Hodge star is characterised by the following property: if $\beta_\tau \in \bigwedge^k \mathcal{E}^1_{\mathbf{C},\tau}$, then for all $\alpha_\tau \in \bigwedge^k \mathcal{E}^1_{\mathbf{C},\tau}$ we have

$$\alpha_\tau \wedge \overline{*_\tau \beta_\tau} = \langle \alpha_\tau, \beta_\tau \rangle_\tau \mathrm{Vol}_\tau.$$

These local maps glue together to form a linear bundle isomorphism $* : \mathcal{E}^k_{\mathbf{C}} \longrightarrow \mathcal{E}^{2-k}_{\mathbf{C}}$.

3.2. **Hodge star operators on modular curves.** Let $X = X_0(N)$ denote the closed modular curve associated to $\Gamma_0(N)$. It is a compact oriented complex manifold of real dimension 2. Recall the Poincaré Hermitian product on $\mathcal{E}^1_{\mathbf{C}}$ induced by the inner product on $\mathcal{E}^1_{\mathbf{R}}$ given by the matrix $\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$ with respect to the basis $\{dx, dy\}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and consider the action of γ on $\mathcal{H}$ which is an isomorphism from $\mathcal{H}$ to itself. This map induces a pull-back map on 1-forms given by $\gamma^* d\tau = (c\tau + d)^{-2} d\tau$ and $\gamma^* d\bar{\tau} = (c\bar{\tau} + d)^{-2} d\bar{\tau}$. Since $\Im(\gamma(\tau)) = y/|c\tau + d|^2$ we see that

$$\langle\, \cdot \,, \cdot\, \rangle_{\gamma(\tau)} = |c\tau + d|^{-4} \langle\, \cdot \,, \cdot\, \rangle_\tau.$$

Using the properties of the Hermitian product we therefore see that

$$\begin{aligned}
\langle \gamma^* d\tau, \gamma^* d\tau \rangle_\tau &= (c\tau + d)^{-2} (c\bar{\tau} + d)^{-2} \langle d\tau, d\tau \rangle_\tau = \langle d\tau, d\tau \rangle_{\gamma(\tau)} \\
\langle \gamma^* d\bar{\tau}, \gamma^* d\bar{\tau} \rangle_\tau &= (c\bar{\tau} + d)^{-2} (c\tau + d)^{-2} \langle d\bar{\tau}, d\bar{\tau} \rangle_\tau = \langle d\bar{\tau}, d\bar{\tau} \rangle_{\gamma(\tau)} \\
\langle \gamma^* d\tau, \gamma^* d\bar{\tau} \rangle_\tau &= 0 = \langle d\tau, d\bar{\tau} \rangle_{\gamma(\tau)}.
\end{aligned}$$

Thus the Poincaré metric is invariant under the action of $\Gamma_0(N)$ and it therefore induces a Hermitian metric on X. The volume form on X associated to this metric is given by the same expression $\mathrm{Vol}_\tau = \frac{dx \wedge dy}{y^2}$ as before. Note that this volume form is invariant under the action of $\Gamma_0(N)$ and therefore defines a 2-form on X.

Let $\mathcal{C}^\infty_{X,\mathbf{R}}$ denote the sheaf of real-valued smooth functions on X and let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Let $\mathcal{E}^1_{X,\mathbf{F}}$ denote the sheaf of $\mathbf{F}$-valued differential 1-forms on X for $\mathbf{F} = \mathbf{R}, \mathbf{C}$. Using the induced Poincaré metric on X we get an induced metric on $\mathcal{E}^2_{X,\mathbf{C}}$ just as in the previous section and similarly we can define the Hodge star

$$* : \mathcal{E}^k_{X,\mathbf{C}} \longrightarrow \mathcal{E}^{2-k}_{X,\mathbf{C}}$$

which is characterised by the following property: if $\beta_\tau \in \bigwedge^k \mathcal{E}^1_{X,\mathbf{C},\tau}$ then for all $\alpha_\tau \in \bigwedge^k \mathcal{E}^1_{X,\mathbf{C},\tau}$ we have

$$\alpha_\tau \wedge \overline{*_\tau \beta_\tau} = \langle \alpha_\tau, \beta_\tau \rangle_\tau \mathrm{Vol}_\tau$$

Remark 3.3. Let x be a non cuspidal point of X. If x is not an elliptic point, then the holomorphic chart on a small enough neighbourhood U of x is given by $\phi : U \longrightarrow \tilde{U}$ where $\tilde{U}$ denotes the open subset of $\mathcal{H}$ in the pre-image of U under the covering map $\mathcal{H} \longrightarrow Y_0(N) := \Gamma_0(N) \backslash \mathcal{H}$ which lies in the fundamental domain of $\Gamma_0(N)$ and the map ϕ is just a lift of the quotient map. Thus for such a point, the curve X is trivially identified with an open subset of $\mathcal{H}$ in the fundamental domain. At elliptic points, one needs to be more careful in defining the local charts, but these are essentially raising to a small power. At cusps, one need to use straightening maps and then wrap-around maps. Note that if for example $36|N$ then there are no elliptic points.

Because of this complex structure on X and the fact that the metric on X is induced from the Poincaré metric on $\mathcal{H}$, one can compute the Hodge star on X locally around non cuspidal, non elliptic points by computing it locally on $\mathcal{H}$ as we did in (1). One needs to be a little more careful around elliptic points and cusps but we will omit this here.

Consider the universal generalized elliptic curve for $\Gamma_0(N)$ given by $\pi : \mathcal{E} \longrightarrow X$ with zero-section $e : X \longrightarrow \mathcal{E}$. Consider the cotangent space at the origin $\underline{\omega} := e^*\Omega^1_{\mathcal{E}/X}$ which is a complex line bundle over X (invertible sheaf of $\mathcal{O}_X$-modules). Denote by $\underline{\omega}^k$ the holomorphic line bundle $\underline{\omega}^{\otimes k}$. It satisfies the property that $M_k(N) \cong H^0(X, \underline{\omega}^k)$ where the isomorphism is given by $f \mapsto \omega_f = f(\tau)(dz)^k$ where z is the local coordinate on the fiber $\pi^{-1}(\tau)$ and dz locally trivialises $\underline{\omega}$. The bundle $\underline{\omega}^k$ is equipped with a Hermitian metric (with logarithmic poles) given in the fibers by $\langle \omega_f, \omega_g \rangle_\tau := f(\tau)\overline{g(\tau)}y^k$. Let $\mathcal{E}^{p,q}$ denote the sheaf of complex-valued differential forms on X of type (p, q). It is equipped with the Hermitian product induced by the Poincaré metric. The bundle $\mathcal{E}^{p,q} \otimes \underline{\omega}^k := \mathcal{E}^{p,q} \otimes_{\mathcal{O}_X} \underline{\omega}^k$ is thus equipped with the Hermitian product

$$\langle \alpha \otimes \omega_f, \beta \otimes \omega_g \rangle_\tau := \langle \alpha, \beta \rangle_\tau \langle \omega_f, \omega_g \rangle_\tau = \langle \alpha, \beta \rangle_\tau f(\tau)\overline{g(\tau)}y^k.$$

Notation. From now on we will simply write $\otimes$ instead of $\otimes_{\mathcal{O}_X}$.

Right exterior product gives a linear isomorphism

$$p : \mathcal{E}^{1-p,1-q}_\tau \otimes (\underline{\omega}^k_\tau)^* \longrightarrow \mathrm{Hom}(\mathcal{E}^{p,q}_\tau \otimes \underline{\omega}^k_\tau, \mathcal{E}^{1,1}_\tau)$$

where $\alpha_\tau \otimes \omega_f \wedge \beta_\tau \otimes \phi := \phi(\omega_f)\alpha_\tau \wedge \beta_\tau$. Moreover, the bundle $\mathcal{E}^{1,1} = \mathcal{E}^2_{\mathbf{C}}$ is trivialised by Vol.

The Riesz representation theorem gives an anti-linear isomorphism

$$m : \mathcal{E}^{p,q}_\tau \otimes \underline{\omega}^k_\tau \longrightarrow (\mathcal{E}^{p,q}_\tau \otimes \underline{\omega}^k_\tau)^*$$

defined by $m(\alpha_\tau \otimes \omega_f) = \langle \cdot\,, \alpha_\tau \otimes \omega_f \rangle_\tau$. We get anti-linear isomorphisms $\bar{*}_{k,\tau} := p^{-1} \circ m$ which glue together to form an anti-linear isomorphism of bundles

$$\bar{*}_k : \mathcal{E}^{p,q} \otimes \underline{\omega}^k \longrightarrow \mathcal{E}^{1-p,1-q} \otimes (\underline{\omega}^k)^*$$

characterised by the following property: if $\beta \otimes \omega_g \in \mathcal{E}^{p,q} \otimes \underline{\omega}^k(U)$ for some open subset U of X, then for all $\tau \in U$ and all $\alpha_\tau \otimes \omega_f \in \mathcal{E}^{p,q}_\tau \otimes \underline{\omega}^k_\tau$ we have

$$\alpha_\tau \otimes \omega_f \wedge (\bar{*}_k(\beta \otimes \omega_g))_\tau = \langle \alpha_\tau \otimes \omega_f, \beta_\tau \otimes \omega_g \rangle \mathrm{Vol}_\tau.$$

By Riesz's representation theorem we have an anti-linear isomorphism

$$r : \underline{\omega}^k_\tau \longrightarrow (\underline{\omega}^k_\tau)^*, \qquad \omega_f \mapsto \langle \cdot\,, \omega_f \rangle_\tau.$$

We have a perfect pairing

$$(\mathcal{E}^{0,0} \otimes_{\mathcal{O}_X} \underline{\omega}^k) \times (\mathcal{E}^{0,0} \otimes_{\mathcal{O}_X} \underline{\omega}^{-k}) \longrightarrow \mathcal{E}^{0,0}$$

given on sections over U by $(s\omega_f, t\omega_g) \mapsto stfg$. This identifies $\mathcal{E}^{0,0} \otimes_{\mathcal{O}_X} \underline{\omega}^{-k}$ with the dual sheaf $(\mathcal{E}^{0,0} \otimes_{\mathcal{O}_X} \underline{\omega}^k)^*$. If $t \in \mathcal{E}^{0,0}(U)$ and g is a modular form of weight k, then we denote the element corresponding to tg in $\mathcal{E}^{0,0} \otimes \underline{\omega}^k(U)$ by ω_{tg}. If $\omega_f \in \underline{\omega}^k(U)$, then $r(\omega_f)$ corresponds to $\omega_{\bar{f}y^k}$ in $\mathcal{E}^{0,0} \otimes_{\mathcal{O}_X} \underline{\omega}^{-k}(U)$.

Proposition 3.4. *With the above identifications, the Hodge star is an anti-linear isomorphism of bundles $\bar{*}_k : \mathcal{E}^{p,q} \otimes \underline{\omega}^k \longrightarrow \mathcal{E}^{1-p,1-q} \otimes \underline{\omega}^{-k}$ given on sections by $\bar{*}_k(\beta \otimes \omega_g) = \overline{*\beta} \otimes \omega_{\bar{f}y^k}$.*

Proof. For all τ and all $\alpha_\tau \otimes \omega_f \in \mathcal{E}^{p,q}_\tau \otimes \underline{\omega}^k_\tau$ we have

$$\begin{aligned}\alpha_\tau \otimes \omega_f \wedge (\overline{*\beta})_\tau \otimes \omega_{\bar{f}y^k} &= \alpha_\tau \wedge (\overline{*\beta})_\tau f(\tau)\overline{g(\tau)}y^k = \alpha_\tau \wedge \overline{*_\tau\beta_\tau}\langle \omega_f, \omega_g \rangle_\tau \\ &= \langle \alpha_\tau, \beta_\tau \rangle_\tau \langle \omega_f, \omega_g \rangle_\tau \mathrm{Vol}_\tau = \langle \alpha_\tau \otimes \omega_f, \beta_\tau \otimes \omega_g \rangle_\tau \mathrm{Vol}_\tau\end{aligned}$$

where in the third equality we used the characterisation of the Hodge star on $\mathcal{E}^{p+q}_{X,\mathbf{C},\tau}$. $\square$

Notation. Let U be an open subset of X. We will use the notation $f\alpha$ to denote an element in $\mathcal{E}^{p,q}\otimes\underline{\omega}^k(U)$. Here, f is some function from $\mathcal{H}$ to $\mathbf{C}$ and α is a $\mathbf{C}$-valued differential form on $\mathcal{H}$ of type (p,q). It is implicitly understood that f should decompose into a product of 3 functions $f=f_1f_2f_3$ where $f_1\in\mathcal{E}^{0,0}(U)$, f_2 has the property that $f_2\alpha\in\mathcal{E}^{p,q}(U)$ and f_3 is holomorphic on U such that $f_3(dz)^k\in\underline{\omega}^k$. Then $f\alpha$ stands for the element

$$f_1\otimes f_2\alpha\otimes f_3(dz)^k\in\mathcal{E}^{0,0}\otimes_{\mathcal{C}^\infty_{X,\mathbf{C}}}\mathcal{E}^{p,q}\otimes\underline{\omega}^k(U)=\mathcal{E}^{p,q}\otimes\underline{\omega}^k(U).$$

We now turn to the computation of the Hodge star $\bar{*}_k$. Let U be an open subset of X. Using the previous proposition together with (1) we get:

- If $f\in\mathcal{E}^{0,0}\otimes\underline{\omega}^k(U)$, then $\bar{*}_k(f)=\bar{f}y^k\overline{*1}=\bar{f}y^{k-2}dx\wedge dy=\frac{i}{2}\bar{f}y^{k-2}d\tau\wedge d\bar{\tau}$.
- If $fd\tau\in\mathcal{E}^{1,0}\otimes\underline{\omega}^k(U)$, then
$$\bar{*}_k(fd\tau)=\bar{f}y^k\overline{*(dx+idy)}=\bar{f}y^k*(dx-idy)=\bar{f}y^k(dy+idx)=i\bar{f}y^kd\bar{\tau}.$$
- If $fd\bar{\tau}\in\mathcal{E}^{0,1}\otimes\underline{\omega}^k(U)$, then
$$\bar{*}_k(fd\bar{\tau})=\bar{f}y^k\overline{*(dx-idy)}=\bar{f}y^k*(dx+idy)=\bar{f}y^k(dy-idx)=-i\bar{f}y^kd\tau.$$
- If $fd\tau\wedge d\bar{\tau}\in\mathcal{E}^{1,1}\otimes\underline{\omega}^k(U)$, then
$$\bar{*}_k(fd\tau\wedge d\bar{\tau})=\bar{f}y^k*(d\bar{\tau}\wedge d\tau)=\bar{f}y^k2i*(dx\wedge dy)=2iy^{2+k}\bar{f}.$$

3.3. **Laplacians.** Note that $\mathcal{E}^{1,1-q}=\mathcal{E}^{0,1-q}\otimes\Omega^1_X$ where Ω^1_X is the canonical bundle of X, that is the bundle of holomorphic differential 1-forms on X. Using the Hodge star

$$\bar{*}_k:\mathcal{E}^{0,q}\otimes\underline{\omega}^k\longrightarrow\mathcal{E}^{0,1-q}\otimes\Omega^1_X\otimes\underline{\omega}^{-k}$$

one can define $\bar{\partial}^*_k:=(-1)^q(\bar{*}_k)^{-1}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\circ\bar{*}_k:\mathcal{E}^{0,q}\otimes\underline{\omega}^k\longrightarrow\mathcal{E}^{0,q-1}\otimes\underline{\omega}^k$ and show that this is the formal adjoint of $\bar{\partial}_k:\mathcal{E}^{0,q}\otimes\underline{\omega}^k\longrightarrow\mathcal{E}^{0,q+1}\otimes\underline{\omega}^k$ with respect to the Hermitian L^2-metric on $H^0(X,\mathcal{E}^{0,q}\otimes\underline{\omega}^k)$ given by $(\alpha,\beta)_{L^2}:=\int_X\langle\alpha,\beta\rangle\mathrm{Vol}$ where $\langle\alpha,\beta\rangle$ is the function $\tau\mapsto\langle\alpha_\tau,\beta_\tau\rangle_\tau$. This metric is defined since X is compact and oriented. We remark that by the characterisation of the Hodge star we have $(\alpha,\beta)_{L^2}=\int_X\alpha\wedge\bar{*}_k\beta$. Using this formal adjoint we can define the Laplacian

$$\Delta_{\underline{\omega}^k}:=\bar{\partial}_k\bar{\partial}^*_k+\bar{\partial}^*_k\bar{\partial}_k:\mathcal{E}^{0,q}\otimes\underline{\omega}^k\longrightarrow\mathcal{E}^{0,q}\otimes\underline{\omega}^k$$

which is an elliptic differential operator of order 2. The interested reader can learn more about the theory of Laplacians in [Voi].

Proposition 3.5. *The Laplacian* $\Delta_{\underline{\omega}^k}:H^0(X,\mathcal{E}^{0,0}\otimes\underline{\omega}^k)\longrightarrow H^0(X,\mathcal{E}^{0,0}\otimes\underline{\omega}^k)$ *is given by*

$$\Delta_{\underline{\omega}^k}=\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\circ\bar{*}_k\circ\bar{\partial}_k=\frac{1}{2}\Delta_k.$$

Proof. The adjoint operator $\bar{\partial}^*_k$ on $\mathcal{E}^{0,0}\otimes\underline{\omega}^k$ is the trivial map. Thus $\Delta_{\underline{\omega}^k}=\bar{\partial}^*_k\circ\bar{\partial}_k$. By Remark 3.1 we have $*^{-1}=*$ on $\mathcal{E}^{0,0}$ and thus we see that $(\bar{*}_k)^{-1}=\bar{*}_{-k}$ on $\mathcal{E}^{0,0}\otimes\underline{\omega}^k$. Thus $\bar{\partial}^*_k=\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\circ\bar{*}_k$. Let $f\in H^0(X,\mathcal{E}^{0,0}\otimes\underline{\omega}^k)$. Using our Hodge star computations above, we see that

$$\begin{aligned}\Delta_{\underline{\omega}^k}f&=\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\circ\bar{*}_k(\bar{\partial}_kf)=\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\circ\bar{*}_k\left(\frac{\partial f}{\partial\bar{\tau}}d\bar{\tau}\right)\\&=\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}\left(-iy^k\overline{\frac{\partial f}{\partial\bar{\tau}}}d\tau\right)=-\frac{1}{2}\bar{*}_{-k}\circ\bar{\partial}_{\Omega^1_X\otimes\underline{\omega}^{-k}}(\xi_kfd\tau)\\&=\frac{1}{2}\bar{*}_{-k}\left(\frac{\partial}{\partial\bar{\tau}}(\xi_kf)d\tau\wedge d\bar{\tau}\right)=iy^{2-k}\overline{\frac{\partial}{\partial\bar{\tau}}(\xi_kf)}\\&=\frac{1}{2}\xi_{2-k}(\xi_kf)=\frac{1}{2}\Delta_k.\end{aligned}$$

□

3.4. **Proof of surjectivity of ξ_{2-k}.** We use the same notation as above. Consider the Dolbeault resolution of X

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \longrightarrow 0.$$

Let $\{s_1, \cdots, s_r\}$ denote the cusps of X and form the divisor $D := \sum_{i=1}^r s_i \in \mathrm{Div}(X)$. Let n be a positive integer and let $\mathcal{O}_{nD}$ denote the holomorphic line bundle attached to the divisor nD, that is the invertible sheaf of $\mathcal{O}_X$-modules whose sections are given by

$$\mathcal{O}_{nD}(U) = \{f \in \mathcal{K}(U) : \mathrm{div}(f) \geq -nD\}$$

where $\mathcal{K}$ is the sheaf of rational functions on X. The tensor product sheaf $\underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1 (a holomorphic line bundle) and thus by tensoring with the Dolbeault resolution we get an exact sequence

$$0 \longrightarrow \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD} \longrightarrow \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD} \xrightarrow{\bar{\partial}\otimes 1 \otimes 1} \mathcal{E}^{0,1} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD} \longrightarrow 0.$$

The sheaves $\mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}$ and $\mathcal{E}^{0,1} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}$ are sheaves of $\mathcal{C}^{\infty}_{X,\mathbf{C}}$-modules. It follows that they are fine sheaves and as a consequence they are acyclic. By taking the long exact sequence in cohomology we therefore get an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow H^0(X, \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow H^0(X, \mathcal{E}^{0,1} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \\ \longrightarrow H^1(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow 0. \end{aligned}$$

We claim that for n large enough, the group $H^1(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$ vanishes. By Serre duality we have an isomorphism

$$H^1(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \cong H^0(X, \Omega^1_X \otimes (\underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})^*)^* \cong H^0(X, \Omega^1_X \otimes \underline{\omega}^{k-2} \otimes \mathcal{O}_{-nD})^*.$$

Recall the Kodaira-Spencer isomorphism $\mathrm{KS} : \underline{\omega}^2 \longrightarrow \Omega^1_X(\log \mathrm{cusps})$ given by $(dz)^2 \mapsto d\tau$. This gives an isomorphism

$$M_{r+2}(N) \cong H^0(X, \underline{\omega}^{r+2}) \cong H^0(X, \underline{\omega}^r \otimes \Omega^1_X(\log \mathrm{cusps}))$$

and we have an identification

$$S_{r+2}(N) \cong H^0(X, \underline{\omega}^r \otimes \Omega^1_X).$$

We therefore have

$$H^0(X, \Omega^1_X \otimes \underline{\omega}^{k-2} \otimes \mathcal{O}_{-nD}) = \{f \in S_k(N) : \mathrm{div}(f) \geq nD\}.$$

Thus if f belongs to this space, then $\deg(\mathrm{div}(f)) \geq nr$. But the degree of $\mathrm{div}(f)$ for $f \in S_k(N)$ only depends on the weight k since if $g \in S_k(N)$, then $f/g \in \mathcal{K}(X)$ so $\deg(\mathrm{div}(f/g)) = 0$ which implies that $\deg(\mathrm{div}(f)) = \deg(\mathrm{div}(g))$. But for $f \in H^0(X, \Omega^1_X \otimes \underline{\omega}^{k-2} \otimes \mathcal{O}_{-nD})$ we have $\deg(\mathrm{div}(f)) \to +\infty$ as $n \to +\infty$ so for n large enough we must have $H^0(X, \Omega^1_X \otimes \underline{\omega}^{k-2} \otimes \mathcal{O}_{-nD}) = 0$. This proves our claim.

Let $g \in M_k^!(N)$. Our goal is to find $f \in H_{2-k}(N)$ such that $\xi_{2-k} f = g$. Choose n a positive integer large enough such that $n \geq \max_{1 \leq i \leq r}\{\mathrm{ord}_{s_i}(g)\}$ and $H^1(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$ vanishes. Then we have a short exact sequence

$$0 \longrightarrow H^0(X, \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow H^0(X, \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow H^0(X, \mathcal{E}^{0,1} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD}) \longrightarrow 0.$$

Since g transforms as a weight k modular form we see that $g d\tau \otimes (dz)^{k-2}$ defines a global $\underline{\omega}^k$-valued differential form on X of type $(1,0)$. In our above notation this form is denoted simply by $g d\tau$. By our choice of n we see that $g d\tau \in H^0(X, \mathcal{E}^{1,0} \otimes \underline{\omega}^{k-2} \otimes \mathcal{O}_{nD})$. Applying the Hodge star operator we get

$$\bar{*}_{k-2}(g d\tau) = i y^{k-2} \bar{g} d\bar{\tau} \in H^0(X, \mathcal{E}^{0,1} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$$

and by the above exact sequence there exists $f \in H^0(X, \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$ such that $\bar{\partial} f = iy^{k-2}\bar{g}d\bar{\tau}$. This implies

$$-iy^{2-k}\frac{\partial f}{\partial \bar{\tau}} = \bar{g} \Rightarrow \xi_{2-k}\left(\frac{f}{2}\right) = g.$$

It remains to be seen that $f/2 \in H_{2-k}(N)$. We have $f \in H^0(X, \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$ so $f/2$ transforms under $\Gamma_0(N)$ as a weight $2-k$ form. By Proposition 3.5 we have

$$\begin{aligned}\Delta_{2-k}\left(\frac{f}{2}\right) &= \Delta_{\underline{\omega}^{2-k}}(f) = \bar{*}_{k-2} \circ \bar{\partial}_{\Omega^1_X \otimes \underline{\omega}^{k-2}} \circ \bar{*}_{2-k} \circ \bar{\partial}_{2-k}(f) = \bar{*}_{k-2} \circ \bar{\partial}_{\Omega^1_X \otimes \underline{\omega}^{k-2}} \circ \bar{*}_{2-k}(iy^{k-2}\bar{g}d\bar{\tau}) \\ &= \bar{*}_{k-2} \circ \bar{\partial}_{\Omega^1_X \otimes \underline{\omega}^{k-2}}(gd\tau) = \bar{*}_{k-2}\left(\frac{\partial g}{\partial \bar{\tau}} d\bar{\tau} \wedge d\tau\right) = 0\end{aligned}$$

since $\frac{\partial g}{\partial \bar{\tau}} = 0$ by holomorphy of g on $\mathcal{H}$. We know that f is modular and is annihilated by the hyperbolic Laplacian of weight $2-k$. Reasoning as in the beginning of these notes, f admits a Fourier expansion of the form

$$f(\tau) = \sum_{n \in \mathbf{Z}} c_f^+(n) q^n + c_f^-(0) y^{k-1} + \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} c_f^-(n) H(2\pi n y) e(nx)$$

and consequently we may speak of the holomorphic and the non-holomorphic parts of f. By Proposition 1.5 and the fact that $\xi_{2-k}(f/2) = g$ is a weakly holomorphic modular form, we must have $c_f^-(n) = 0$ for all but finitely many $n > 0$. This must hold at all cusps. Finally, since $f \in H^0(X, \mathcal{E}^{0,0} \otimes \underline{\omega}^{2-k} \otimes \mathcal{O}_{nD})$ we see that the holomorphic part of f at each cusp must have finite tail. This proves that $f/2$ satisfies the required growth conditions at the cusps and thus $f/2 \in H_{2-k}(N)$ as desired.

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